Pseudocycles and Smooth Intersection Theory

Bachelorarbeit

zur Erlangung des akademischen Grades
Bachelor of Science (B.Sc.)

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Eingereicht am Institut für Mathematik der Humboldt-Universität zu Berlin am:

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1 Introduction

The subject of this thesis, pseudocycles, can be understood as a way of conceptualizing singular homology with integer coefficients, complete with intersection product, in the case of smooth manifolds. They are an interesting alternative perspective on homology, since elements in the pseudocycle groups of a manifold M are represented by smoooth maps from a smooth oriented manifold without boundary into M with the definition of the intersection product relying solely on the theory of smooth manifolds and not needing any algebraic topology. So we get a theory that is somewhat closer to our intuition, but isomorphic to integral homology according to the results of [Zin08].

Before we introduce pseudocycles themselves, it will be in order to talk about some preliminaries and general concepts from differential topology which will turn out to have special relevance to the theory of pseudocycles: The concept of *transversality* in section 2.1, and the idea of associating *orientations* to smooth manifolds via their tangent spaces in section 2.2. After these concepts are established, section 3 will start with the basic definitions of pseudocycle theory and the computation of the zero-dimensional pseudocycle groups of any smooth manifold, before turning to the intersection product in 3.2 and then using this new tool to compute the 1-dimensional pseudocycle group of the torus $\mathbb{T}^2$ in section 3.3.

Finally, in section 3.4.1 we will describe how every integral homology class can be presented as a bordism class of pseudocycles via a natural homomorphism respecting the intersection product, and in section 3.4.2 we will sketch the rough idea of the reverse homomorphism.

Remark. All manifolds spoken of in this thesis are assumed to be smooth and paracompact. A manifold M is called **paracompact** if every open cover $M \subseteq \bigcup_{i \in I} U_j$ has a locally finite open refinement. An **open refinement** of an open cover is a new open cover $M \subseteq \bigcup_{j \in J} V_j$ such that for any $j \in J$, there exists $i \in I$ such that $V_j \subseteq U_i$. And an open cover is said to be **locally finite** if every point in M has a neighborhood which intersects only finitely many sets in the cover.

When a homology group $H_d(M)$ of a space M is mentioned in this thesis, it is always implied that we are speaking about homology with integer coefficients $H_d(M; \mathbb{Z})$. Maps between manifolds are assumed to be smooth, unless otherwise specified.

2 Transversality and Orientations

2.1 Transversality

Transversality is going to be crucial in the definition of the intersection product on pseudocycle groups, which will be discussed in section 3.2. As we will see, transversality is the natural generalization of the idea of a regular value to submanifolds of dimensions greater than zero.

Definition 2.1. *Suppose X, Y are manifolds. A smooth map $f : X \to Y$ is called **transverse** to a submanifold $Z \subset Y$ if the following holds for all $x \in f^{-1}(Z)$:*

$$\mathrm{Im}(df_x) + T_y(Z) = T_y(Y).$$

We write $f \pitchfork Z$.

The meaning of transversality becomes clear when we consider the following central theorem:

Theorem 2.1. *If X, Y are manifolds and $f : X \to Y$ a smooth map which is transverse to a submanifold $Z \subset Y$, then its preimage $f^{-1}(Z)$ is a submanifold of X, and the codimension of $f^{-1}(Z)$ in X is equal to the codimension of Z in Y.*

Proof. For any $x \in f^{-1}(Z)$, Z is cut out by l independent functions $(g_i : Y \to \mathbb{R})_{1\leq i\leq l}$ in an open neighborhood U of $y = f(x)$ (cf. [GP74], p. 24); that is, there is a collection of l independent maps $g_i : U \to \mathbb{R}$ (where l is the codimension of Z in Y) such that $Z \cap U$ is the zero set of the maps $g_1, \ldots, g_l$. Combining these maps into one, we define a map $g : U \to \mathbb{R}^l$ as the submersion $(g_1, \ldots, g_l)$.

Now, if 0 should turn out to be a regular value of $(g \circ f) : X \to \mathbb{R}^l$, the "Preimage Theorem", which is a direct consequence of the inverse function theorem (cf. [GP74], p. 21) would guarantee that $(g \circ f)^{-1}(0)$ is a submanifold with codimension l in X.

Zero is a regular value of $(g \circ f)$ precisely if $d(g \circ f)_x = dg_y \circ df_x : T_x(X) \to \mathbb{R}^l$ is surjective. We know that dg_y is a surjective linear transformation, and its kernel is precisely $T_y(Z)$ (cf. [GP74], 24). So if, as our definition of transversality demands, $\mathrm{Im}(df_x)$ is at least as big as the complement of $T_y(Z)$ in $T_y(Y)$, we can rest assured that $d(g \circ f)_x$ is surjective, and thus that 0 is a regular value, and therefore that $f^{-1}(Z)$ is a submanifold with codimension l. □

Remark. If Z is a single point z, we will have $T_y(Z) = 0 \subset T_y(Y)$ and thus $f \pitchfork z$ exactly if $df_x : T_x(X) \to T_y(Y)$ is surjective, which is to say that z is a regular value of f. So regular values are just a special case of transverse submanifolds.

In the case where X is a manifold with boundary, the following theorem holds, which is mentioned without proof here; for details on the proof, we refer to page 61 of [GP74]:

Theorem 2.2. *If X is a manifold with boundary, Y a manifold without boundary, and $f : X \to Y$ a smooth map with both $f : X \to Y$ and $\partial f : \partial X \to Y$ (where $\partial f := f_{|\partial X}$) transverse to a submanifold without boundary $Z \subset Y$, then $f^{-1}(Z)$ is a manifold with boundary,*

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X,$$

and the codimension of $f^{-1}(Z)$ in X is equal to the codimension of Z in Y.

□

Building on this definition of transversality as a relation between a map and a submanifold, this notion can be extended to two submanifolds $X \subset Y$ and $Z \subset Y$, by simply considering the inclusion map $i_X : X \hookrightarrow Y$.

A point $x \in X$ is in the preimage $i_X^{-1}(Z)$ if and only if $x \in X \cap Z$, and the derivative $d(i_X)_x : T_x(X) \to T_x(Y)$ is just the inclusion of $T_x(X)$ into $T_x(Y)$. Thus $i_X \pitchfork Z$ if and only if for all $x \in X \cap Z$ the following equation holds:

$$T_x(X) + T_x(Z) = T_x(Y) \tag{2.1}$$

Definition 2.2. *Two submanifolds $X \subset Y$ and $Z \subset Y$ are called* ***transverse****, if (2.1) holds for all $x \in X \cap Z$. We write $X \pitchfork Z$*

An application of Theorem 2.1, together with a few arithmetic transformations gives us:

Theorem 2.3. *The intersection of two transverse submanifolds $X \subset Y$ und $Z \subset Y$ is itself a submanifold, and*

$$Codim(X \cap Z) = Codim(X) + Codim(Z)$$

□

Remark. Two submanifolds whose dimensions are in sum not at least equal to the dimension of the ambient manifold can only be transverse, according to our definition, if their intersection is empty.

So we now know what it means for a map and a submanifold to be transverse, and what it means for two submanifolds to be transverse; there is one more notion of transversality that we are going to need, and that is transversality of two maps.

Definition 2.3. *For two maps $f : X \to M$ and $g : Y \to M$ with the same manifold as their codomain, we say that f and g are* ***transverse*** *(and we write $f \pitchfork g$) if the map $f \times g : X \times Y \to M \times M$ is transverse to the diagonal $\Delta \subset M \times M$.*

Another perspective on transversality of two maps, which works without mentioning the diagonal, is given by the following Lemma.

Lemma 2.4. *Two maps $f : X \to M$ and $g : Y \to M$ are transverse if and only if for all $(x, y) \in X \times Y$ such that $f(x) = g(y) =: p$, the following holds:*

$$\mathrm{Im}(df_x) + \mathrm{Im}(dg_y) = T_p(M)$$

Proof. Say that the manifold M is of dimension k, X is of dimension m, Y is of dimension n and that $z = (x, y) \in X \times Y$ is a point such that $f(x) = g(y) =: p$, and let $q := (p, p) \in M \times M$.

First, consider the intersection $\mathrm{Im}(d(f \times g)_z) \cap T_q(\Delta)$. It consists precisely of the points $w \in T_q(M \times M)$ such that there is $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ with $df_x(u) = dg_y(v) = w$. So there is an isomorphism between $\mathrm{Im}(d(f \times g)_z) \cap T_q(\Delta)$ and $\mathrm{Im}(df_x) \cap \mathrm{Im}(dg_y)$.

Then, notice that $\dim(\mathrm{Im}(df_x)) + \dim(\mathrm{Im}(dg_y)) = \dim(\mathrm{Im}(df_x) \times \mathrm{Im}(dg_y))$, and that $\dim(T_q(\Delta)) = k$. So

$$\dim(\mathrm{Im}(df_x)) + \dim(\mathrm{Im}(dg_y)) = \dim(T_p(M)) \tag{2.2}$$

$$\iff \dim(\mathrm{Im}(df_x) \times \mathrm{Im}(dg_y)) + \dim(T_q(\Delta)) = \dim(T_q(M \times M)) \tag{2.3}$$

Putting this together and using some of the things we know from Linear Algebra, we get:

$$\begin{aligned}
& \dim(\mathrm{Im}(d(f\times g)_z)+T_q(\Delta)) = 2k \quad (\text{i.e. } f \pitchfork g) \\
\iff & \underbrace{\dim(\mathrm{Im}(d(f\times g)_z))}_{=\dim(\mathrm{Im}(df_x))+\dim(\mathrm{Im}(dg_y))} + k - \underbrace{\dim(\mathrm{Im}(d(f\times g)_z)\cap T_q(\Delta))}_{=\dim(\mathrm{Im}(df_x)\cap\mathrm{Im}(dg_y))} = 2k \\
\iff & \dim(\mathrm{Im}(df_x)) + \dim(\mathrm{Im}(dg_y)) - \dim(\mathrm{Im}(df_x)\cap\mathrm{Im}(dg_y)) = k \\
\iff & \dim(\mathrm{Im}(df_x)\times\mathrm{Im}(dg_y)) + \dim(T_q(\Delta)) - \dim(\mathrm{Im}(df_x)\cap\mathrm{Im}(dg_y)) = 2k \quad (\text{by } (2.2)) \\
\iff & \dim(\mathrm{Im}(df_x)) + \dim(\mathrm{Im}(dg_y)) - \dim(\mathrm{Im}(df_x)\cap\mathrm{Im}(dg_y)) = k \\
\iff & \dim(\mathrm{Im}(df_x)+\mathrm{Im}(dg_y)) = k
\end{aligned}$$

which means $\mathrm{Im}(df_x)+\mathrm{Im}(dg_y)$ is a k-dimensional subspace of $T_p(M)$, i.e. is the entire space $T_p(M)$.

□

Lemma 2.5. *If $f_1 : X_1 \to M$ and $f_2 : X_2 \to M$ are transverse, then*

$$\{(x,y)\in X_1\times X_2 \mid f_1(x)=f_2(y)\}\subset X_1\times X_2$$

is a smooth submanifold in $X_1\times X_2$ of dimension $dim(X_1)+dim(X_2)-dim(M)$.

Proof. This is a straightforward application of Theorem 2.1 with Z being $\Delta\subset M\times M$ and $f=f_1\times f_2 : X_1\times X_2\to M\times M$. □

We will discuss one more interesting quality of transversality, namely that it is a *generic* quality: For any smooth map $f : X\to Y$ and any submanifold $Z\subset Y$, it is possible to deform f by an arbitrarily small amount to make it transverse to Z. To be able to say what this means, we will define the notions of *meager* and *comeager* subsets in the following way:

Definition 2.4. *A subset S of a topological space X is called* ***meager*** *if it is a countable union of nowhere dense subsets (i.e. sets whose closure has empty interior).*

Definition 2.5. *A subset S of a topological space X is called* ***comeager*** *or* ***residual*** *if it is the complement of a meager set (or, equivalently: if it contains a countable intersection of open dense sets).*

We are also going to introduce the symbol

$$\pitchfork^r(X,Y;Z) := \{f\in C^r(X,Y) : f\pitchfork Z\}$$

to denote all the C^r-maps from X to Y which are transverse to Z, where X, Y and $Z\subset Y$ are manifolds.

To make the notion of deforming by an arbitrarily small amount precise, we will furthermore have to introduce a topology on the set of smooth maps $C^\infty(X,Y)$. The topology we are going to choose is the so called *strong* topology. We will first describe topologies on the spaces $C^r(X,Y)$ (and denote the resulting topological spaces by $C^r_S(X,Y)$) and then define the strong topology on $C^\infty(X,Y)$ to be the union of the topologies induced by the inclusion maps $C^\infty(X,Y)\hookrightarrow C^r(X,Y)$.

So, let us now define a base for the strong topology on $C^r(X,Y)$:

Let $\Phi=\{\phi_i,U_i\}_{i\in J}$ be a *locally finite* set of charts on X, i.e. charts such that every point in M has a neighborhood that has nonempty intersection with only finitely many of the $U_i's$. (We can always find such a set of charts because we assumed X to be paracompact.)

Let $K = \{K_i\}_{i \in J}$ be a family of compact sets such that for every $i \in J$, $K_i \subset U_i$.
Let $\Psi = \{\psi_i, V_i\}_{i \in J}$ be a set of charts on Y.
And let $\epsilon = \{\epsilon_i\}_{i \in J}$ be a family of positive numbers.
Now, for $f \in C^r(X, Y)$ such that $f(K_i) \subset V_i$, a neighborhood of f

$$\mathcal{N}(f; \Phi, \Psi, K, \epsilon)$$

is defined to be the set of maps $g \in C^r(X, Y)$ such that for all $i \in J, x \in \phi_i(K_i)$ and $0 \leq k \leq r$:

$$g(K_i) \subset V_i$$

and

$$\|d^k(\psi_i \circ f \circ \phi_i^{-1})_x - d^k(\psi_i \circ g \circ \phi_i^{-1})_x\| < \epsilon_i.$$

The strong topology on $C^r(X, Y)$ is generated by these sets as a base. And as mentioned before, the strong topology on $C^\infty(X, Y)$ is defined to be the union of the topologies induced by the maps $C^\infty(X, Y) \hookrightarrow C^r(X, Y)$. (Cf. [Hir94] p. 35 and 74 on this.)

We can now state the **Transversality Theorem** (cf. [Hir94], p. 74):

Theorem 2.6. *Let X, Y be smooth manifolds and $Z \subset Y$ a submanifold. Then $\pitchfork^r(X, Y; Z)$ is residual (and therefore dense) in $C^r(X, Y)$ for the strong topology.*

Proof. Cf. §3.2 of [Hir94]. □

A simpler version, which can be proven here, is the *parametric transversality theorem*:

Theorem 2.7. *If X and S are smooth manifolds, and $F : X \times S \to Y$ is a smooth map of manifolds (i.e. a map such that we get a smoothly varying family of maps $f_s(x) = F(x, s)$ indexed by a parameter s ranging over S), where all involved manifolds have empty boundary, and if Z is a submanifold of Y without boundary, and F is transverse to Z, then the set*

$$\pitchfork(F; Z) := \{s \in S : F_s \pitchfork Z\}$$

is residual (and therefore dense).[1]

Proof. Let $W := F^{-1}(Z) \subset X \times S$. Since $F \pitchfork Z$, W is a submanifold. Let $\pi : X \times S \to S$ be the projection. We want to show that $f_s \pitchfork Z$ holds whenever $s \in S$ is a regular value of the restriction of the projection to W, $\pi_{|W} : W \to S$. For if that is the case, Sard's Theorem (for example in the version presented in §3.1 of [Hir94], there called "Morse-Sard Theorem") says that the set of such s is residual. (Which implies also that it is dense.)

So, assume $f_s(x) = z \in Z$. This is the same as saying $F(x, s) = z$, and since $F \pitchfork Z$ we therefore know that

$$dF_{(x,s)}T_{(x,s)}(X \times S) + T_z(Z) = T_z(Y)$$

and so for an arbitrary vector $a \in T_z(Y)$, a vector $b \in T_{(x,s)}(X \times S)$ exists such that

$$dF_{(x,s)}(b) - a \in T_z(Z).$$

To show $f_s \pitchfork_x Z$ (that is, $d(f_s)_x(T_x(X)) + T_z(Z) = T_z(Y)$), we now need to find a vector $v \in T_x(X)$ such that $df_s(v) - a \in T_z(Z)$.

[1]For this theorem and its proof, cf. [GP74], p. 68 f.

We use the following fact about tangent spaces:

$$T_{(x,s)}(X \times S) = T_x(X) \times T_s(S)$$

(this is exercise 1.2.9 in [GP74]) to write $b = (b_X, b_S)$ for vectors $b_X \in T_x(X)$ and $b_S \in T_s(S)$.

In the case $b_S = 0$ we immediately get

$$dF_{(x,s)}(b_X, 0) = df_s(b_X)$$

since the restriction of F to $X \times \{s\}$ is exactly f_s.

In the case $b_S \neq 0$, we consider the projection. We know that

$$d\pi_{(x,s)} : T_x(X) \times T_s(S) \to T_s(S)$$

is just the projection of vector spaces (this is, again, exercise 1.2.9 in [GP74]). And under the assumption that $s \in S$ is a regular value of $\pi_{|W}$,

$$d\pi_{(x,s)} : T_{(x,s)}(W) \to T_s(S)$$

is surjective, so there is a vector of the form (w, b_S) in $T_{(x,s)}(W)$. Because F maps W to Z, $dF_{(x,s)}(w, b_S) \in T_z(Z)$, so we can take $v = b_x - w \in T_x(X)$ to be the desired vector, since in this case:

$$df_s(v) - a = dF_{(x,s)}((b_X, b_S) - (w, b_s)) - a = \underbrace{dF_{(x,s)}(b_X, b_S) - a}_{\in T_z(Z)} - \underbrace{dF_{(x,s)}(w, b_S)}_{\in T_z(Z)}$$

This concludes the proof. □

2.2 Orientations on Smooth Manifolds

2.2.1 Basic Idea

(This entire section is based on [GP74], §3.2). If we consider a finite-dimensional vector space V and two ordered bases for that space, $\beta = \{v_1, \ldots v_n\}$ and $\tilde{\beta} = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$, there is a unique vector space isomorphism $A : V \to V$ such that $\tilde{\beta} = A\beta$. This allows us to define an equivalence relation on the set of bases of V, namely by considering the determinant of the base change isomorphism. If $\det A > 0$ we will say that β and $\tilde{\beta}$ have the same orientation, if $\det A < 0$ that they have opposite orientation. Because $\det(AB) = \det(A)\det(B)$, this indeed defines an equivalence relation partitioning the set of ordered basis of V into two equivalence classes. An *orientation* is now an arbitrary choice to call one equivalence class *positively oriented* and the other *negatively oriented*. (Choosing one oriented basis and calling it positively oriented or negatively oriented determines the orientation of the vector space completely. For example, for standard Euclidean space we will give it a standard orientation by calling the standard oriented basis positively oriented.)

Remark. As one easily verifies through the basic properties of the determinant, switching two vectors in an ordered basis, or multiplying one vector in the basis by a negative number, produces a basis with opposite orientation.

Building on this definition of the orientation of a vector space, we can define an orientation of a smooth manifold X as a smooth choice of orientations for all the tangent spaces $T_x(X)$. That is, a choice of orientations such that at every $x \in X$ there is a local parametrization $\phi : U \to X$ such that $d\phi_u : U \to T_{\phi(u)}(X)$ is orientation-preserving at every point $u \in U$.

Clearly this will not work in the case of zero-dimensional manifolds (i.e. discrete sets of points); but in this case, we can simply assign an orientation (+) or (−) to each point $x \in X$ without having to worry about smoothness.

An *oriented manifold* is a manifold together with a fixed smooth orientation; reversing the orientation on an oriented manifold X will produce another oriented manifold, which we will denote by $-X$.

2.2.2 Induced Orientations

We will now discuss some ways in which manifolds that arise from oriented manifolds by taking the product, the transverse intersection, the boundary, or the preimage of a transverse map can be assigned orientations depending on the orientations of the original manifolds in a natural way.

Direct Sum and Product Orientation If a vector space $V = V_1 \oplus V_2$ is the direct sum of two oriented vector spaces, we define the induced **direct sum orientation** on V as follows: Let β_1 and β_2 be ordered bases for V_1 and V_2 respectively, and $\beta = (\beta_1, \beta_2)$ the combined ordered basis for V; then we say that

$$\operatorname{sign}(\beta) := \operatorname{sign}(\beta_1)\operatorname{sign}(\beta_2).$$

This lets us define an orientation on the product of two oriented manifolds X and Y as well, since for every point $(x, y) \in X \times Y$:

$$T_{(x,y)} = T_x(X) \times T_y(Y)$$

and we can identify $T_x(X) \times T_y(Y)$ with the direct sum $T_x(X) \oplus T_y(Y)$ (for a vector $v_i \in \beta_1$, define a vector $\tilde{v}_i$ as

$$\tilde{v}_i = (v_i, \underbrace{0, \ldots, 0}_{\dim Y \text{ zeros}})$$

to get a vector in the basis for $T_x(X) \times T_y(Y)$; and we can similarly define $\tilde{w}_i := (0, w_i)$ for vectors $w_i \in \beta_2$.) So defining a direct sum orientation has also given us a notion of **product orientation**. For the ordered basis $(\beta_1 \times 0, 0 \times \beta_2)$ for $T_x(X) \times T_y(Y) = T_{(x,y)}(X \times Y)$, we set its sign to be $\operatorname{sign}(\beta_1)\operatorname{sign}(\beta_2)$.

Remark. Note that the induced orientation on $X \times Y$ will be opposite to the induced orientation on $Y \times X$ if and only if X and Y are both of odd dimension (in this case the two ordered bases (β_1, β_2) and (β_2, β_1) turn into each other by an odd number of switches of vectors, so again by the basic property of the determinant by which each switch of matrix columns flips its sign, in this case these bases will have opposite orientation).

Intersection Orientation (This paragraph is based on [Wen19], p. 358.) If $V_1, V_2 \leq V$ are oriented linear subspaces of a vector space V such that $V_1 + V_2 = V$, with $\operatorname{codim} V_1 = k$ and $\operatorname{codim} V_2 = l$, then $\operatorname{codim}(V_1 \cap V_2) = k + l$, so we can select a basis for $V_1 \cap V_2$:

$$\beta = (a_1, \ldots, a_{n-(k+l)}).$$

If $V_1 \subset V_2$ or vice versa, this basis will be a basis of V_1 (or V_2) and thus already have an orientation; otherwise, there are vectors $v_1^{(1)}, \ldots, v_l^{(1)}$ in V_1 and $v_1^{(2)}, \ldots v_k^{(2)}$ in V_2 such that

$$\beta_1 = (a_1, \ldots, a_{n-(k+l)}, v_1^{(1)}, \ldots, v_l^{(1)}) \text{ is positively oriented as a basis of } V_1$$

$$\beta_2 = (a_1, \ldots, a_{n-(k+l)}, v_1^{(2)}, \ldots v_k^{(2)}) \text{ is positively oriented as a basis of } V_2$$

and

$$\beta_V = (a_1, \dots, a_{n-(k+l)}, v_1^{(1)}, \dots, v_l^{(1)}, v_1^{(2)}, \dots v_k^{(2)}) \text{ is a basis of } V.$$

Which has a sign $\text{sign}(\beta_V)$ as a basis of the oriented vector space V, and we set

$$\text{sign}(\beta) := \text{sign}(\beta_V).$$

Remark. The so induced orientation on $V \cap W$ will be opposite to that on $W \cap V$ if and only if both V and W have odd codimensions.

If X and Z are transverse submanifolds of Y, then $T_x(X) + T_x(Z) = T_y(Y)$, and we can define an orientation on the manifold $X \cap Z$ in this way, since $T_x(X \cap Z) = T_x(X) \cap T_x(Z)$.

Boundary Orientation If X is an oriented manifold of dimension n, then ∂X will be a manifold without boundary of dimension $n-1$. To define an orientation on ∂X, consider a local parametrization $\phi : U \to X$ around a point $x \in \partial X$ (where $U \overset{\text{open}}{\subseteq} \mathbb{H}^n$) and assume $\phi(0) = x$. Then $d\phi_0 : \mathbb{R}^n \to T_x(X)$ is an isomorphism, and we can define the upper half space $H_x(X) \leq T_x(X)$ to be the image of $\mathbb{H}^n$ under $d\phi_0$. This is independent of the choice of parametrization.

The codimension of $T_x(\partial X)$ in $T_x(X)$ is one – so there are exactly two unit vectors in $T_x(X)$ that are orthogonal to $T_x(\partial X)$. One lies in $H_x(X)$, and the other one does not. Call the latter one the *outward unit normal vector* to the boundary, denoted by n_x. We now give $T_x(\partial X)$ an orientation by declaring for an ordered basis $\beta = (v_1, \dots, v_{n-1})$ for $T_x(X)$ that $\text{sign}(\beta)$ is the sign of $(n_x, \beta) = (n_x, v_1, \dots, v_{n-1})$ as a basis for $T_x(X)$ (on which an orientation is already in place).

An example that is going to have some relevance in section 3.1 is the boundary of the product space of a manifold X with the unit interval I, so let's have a look:

Example. For a manifold without boundary X, $I \times X$ is a manifold with boundary, and for all $t \in I$, $X_t := \{t\} \times X$ is naturally diffeomorphic to X. The boundary of $I \times X$ is $X_1 \amalg X_0$. So what are the induced orientations on the two disconnected boundary components? For $x \in X_1$, the outward-oriented normal vector $n_{(1,x)}$ is

$$(1, 0) \in T_1(I) \times T_x(X).$$

Ordered bases of $T_{(1,x)}(X_1)$ have the form $(0 \times \beta)$ (where β is an ordered basis of $T_x(X)$). So our definition of the boundary orientation says that $\text{sign}(0 \times \beta)$ as a basis of $T_{(1,x)}(X_1)$ is the sign of $(n_{(1,x)}, 0 \times \beta) = (1 \times 0, 0 \times \beta)$ as a basis of $T_{(1,x)}(I \times X)$.

Our definition of the product orientation says that $\text{sign}(1 \times 0, 0 \times \beta) = \text{sign}(1)\,\text{sign}(\beta) = \text{sign}(\beta)$; so the boundary orientation agrees with the orientation of X_1 as a copy of X.

How about X_0? For $x \in X_0$, the outward pointing normal vector is $n_{(0,x)} = (-1, 0)$. So, as a boundary orientation, the sign associated to the basis $0 \times \beta$ for $T_{(0,x)}(X_0)$ is

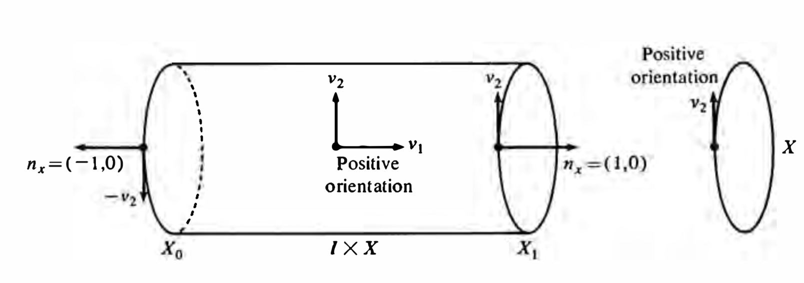
$$\text{sign}(-1 \times 0, 0 \times \beta) = \text{sign}(-1)\,\text{sign}(\beta) = -\,\text{sign}(\beta)$$

– so on X_0, the induced boundary orientation is opposite to the orientation of X_0 as a copy of X. So:

$$\partial(I \times X) = X_1 \amalg -X_0$$

(We will not write "$\partial(I \times X) = X_1 - X_0$" as some authors do, but reserve the notation $A - B$ for two sets A, B for denoting the subset of A consisting of points that are not in B).

Figure 1



Source: [GP74], p. 98

Remark. In the case $\dim(X) = 1$ (and consequently $\dim(\partial X) = 0$) the orientation of a point $x \in \partial X$ will simply be the sign of (n_x) as a basis of $T_x(X)$.

Preimage Orientation Let $f : X \to Y$ be a smooth map with $f \pitchfork Z$, where X, Y, Z are oriented manifolds and $Z \subset Y$.

We now want to define an orientation on the manifold $S := f^{-1}(Z)$. To do this, consider the following: If $f(x) = z \in Z$, then $T_x(S)$ is the preimage of $T_z(Z)$ under the linear transformation $df_x : T_x(X) \to T_z(Y)$ (exercise 1.5.5 in [GP74]). Now choose any subspace H in $T_x(X)$ such that

$$H \oplus T_x(S) = T_x(X)$$

(an obvious choice would be for example the orthogonal complement of $T_x(S)$ in $T_x(X)$, $N_x(S; X)$). Now, by the rules of the direct sum orientation, a choice of orientation for H would immediately also determine an orientation on $T_x(S)$, since for ordered bases $\beta_H, \beta_X, \beta_X$ for the three involved spaces,

$$\operatorname{sign}(\beta_H)\operatorname{sign}(\beta_S) = \operatorname{sign}(\beta_X) \iff \operatorname{sign}(\beta_S) = \operatorname{sign}(\beta_H)\operatorname{sign}(\beta_X)$$

So the only question that remains is: How do we choose an orientation for H? Because of transversality, we have

$$df_x T_x(X) + T_z(Z) = T_z(Y)$$

and, because of how we chose H, and the fact that $df_x^{-1}(T_z(Z)) = T_x(S)$,

$$df_x H \oplus T_z(Z) = T_z(Y)$$

so that we can again use the rules of the direct sum orientation to derive an orientation on $df_x H$ from the orientations of $T_z(Z)$ and $T_z(Y)$, which in turn gives us an orientation on H over the isomorphism df_x.

In the case where $\dim X + \dim Z = \dim Y$, we will have $\dim f^{-1}(Z) = 0$, so it will be a finite number of points. For a point $x \in f^{-1}(Z)$ (with $f(x) = z \in Z$), the fact that dimensions add up, together with $f \pitchfork Z$ gives a direct sum

$$df_x T_x(X) \oplus T_z(Z) = T_z(Y).$$

Now, the orientation of x is positive if the orientations of $df_x T_x(X)$ and $T_z(Z)$ (in that order) "add up" to the orientation on $T_y(Y)$, and negative if not.

3 Pseudocycles

3.1 Pseudocycles and Bordism

Before we can introduce the concept of a pseudocycle, we need the notion of the *omega-limit set* of a smooth map.

Definition 3.1. *The* ***omega-limit set*** *of a smooth map $f : V \to M$ defined on a (possibly noncompact) manifold V is*

$$\Omega_f := \{\lim f(x_n) \mid \text{ sequences } (x_n)_{n\in\mathbb{N}} \subset V \text{ with no limit points}\} \subset M$$

With this definition established, we can define what a *pseudocycle* is:

Definition 3.2. *Let M be a smooth manifold, and V a smooth, oriented manifold of dimension d, with $\partial V = \emptyset$. Let $f : V \to M$ be a smooth map such that the closure of its image, $\overline{f(V)} \subset M$ is compact and $\dim \Omega_f \leq d-2$ (i.e. Ω_f is contained in a countable union of images of smooth maps whose domains are manifolds of dimension at most $d-2$). Then $f : V \to M$ is called a* ***d-dimensional pseudocycle*** *in M.*

We want to use the idea of pseudocycles to define a functor from the category of smooth manifolds to the category of abelian groups that is similar (hopefully isomorphic) to integral homology. As it stands, there are way too many pseudocycles. We will therefore now define an equivalence relation on this set, the notion of *bordism* between pseudocycles. Two d-dimensional pseudocycles shall be called *bordant* if an appropriate $(d+1)$-dimensional smooth oriented manifold with boundary exists such that its boundary is the disjoint union of the two pseudocycles with specified appropriate orientations, and we can define a smooth map on it whose codomain is M and which restricts to the two individual pseudocycle maps on the boundary:

Definition 3.3. *A* **bordism** *between two d-dimensional pseudocycles $f_+ : V_+ \to M$ and $f_- : V_- \to M$ is a smooth map $f : V \to M$ where V is a smooth oriented manifold of dimension $d+1$ such that $\partial V = (-V_-) \amalg V_+$ and $f\mid_{V_\pm} = f_\pm$, the closure of its image, $\overline{f(V)} \subset M$ is compact and the dimension of its omega-limit Ω_f is at most $d-1$.*

Lemma 3.1. *The bordism-relation is an equivalence relation.*

Proof. Clearly the definition of bordism is symmetric.

Every pseudocycle $f_1 : V_1 \to M$ is bordant to itself by the bordism $V = I \times V_1$ with $f : I \times V_1 \to M$ given by $f(t,v) = f(v)$.

If $f_1 : V_1 \to M$ is bordant to $f_2 : V_2 \to M$ via a manifold $V^{(1)}$, and f_2 is bordant to $f_3 : V_3 \to M$ via $V^{(2)}$, then a bordism between f_1 and f_3 can be constructed by gluing $V^{(1)}$ to $V^{(2)}$ along V_2. The map $f : \left(V^{(1)} \cup V^{(2)}\right)/V_2 \to M$ given by the two bordism maps on the two components will be continuous; and by Theorem 2.6 in [Hir94], there exists a smooth map arbitrarily close to f in a neighborhood of V_2 in the strong topology, and one will be able to construct a smooth map that agrees with f_1 and f_2 respectively on the boundary. □

Definition 3.4. *We can now define the* ***d-dimensional pseudocycle-group*** *$H_d^\Psi(M)$ as follows:*

$$H_d^\Psi(M) = \{d\text{-dimensional pseudocycles in } M\}/\sim$$

Where $(f_1 : V_1 \to M) \sim (f_2 : V_2 \to M)$ if there exists a bordism between f_1 and f_2. On this set, the structure of an abelian group can be defined by taking disjoint union to be addition and the empty pseudocycle (that is, the pseudocycle with $V = \emptyset$) to be the neutral element.

Lemma 3.2. *For every pseudocycle $f : V \to M$, the inverse element of $[f] \in H_d^\Psi(M)$ is represented by the same map $f : -V \to M$ with the orientation of the domain reversed.*

Proof. Define $\tilde{V} = V \times I$ and $f : \tilde{V} \to M$ with $f(v,x) = f(v)$. Independently of the orientation we choose for $\tilde{V}$, the induced orientation on the two non-empty connected components of its boundary (which are two copies of V) will be the orientation of V on one and the opposite orientation on the other (cf. section 2.2.2).

$$\partial\tilde{V} = V \amalg -V \amalg \emptyset = (V \amalg -V) \amalg \emptyset$$

Thus we have contructed a boridsm between $V \amalg -V$ and the empty pseudocycle. □

Lemma 3.3. *For every smooth connected manifold M,*

$$H_0^\Psi(M) \cong \mathbb{Z};$$

or, more generally, for every smooth manifold M:

$$H_0^\Psi(M) \cong \bigoplus_{\pi_0(M)} \mathbb{Z}$$

where $\pi_0(M)$ denotes the set of path components of M.

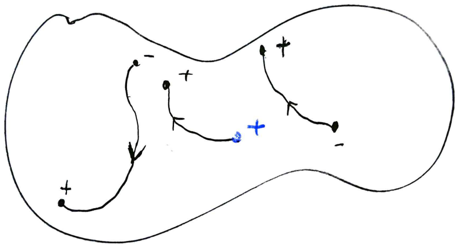
Proof. The pseudocycles of dimension zero are just the maps $f : V \to M$ where V is a compact oriented 0-manifold, i.e. a finite set of discrete points, to each of which a sign $(+)$ or $(-)$ has been associated. The other criteria mentioned in the definition of pseudocycles are trivially true for any such manifold, since manifolds of negative dimension are always empty.

For every path component M_i of M, we can now define a homomorphism ϕ_i from the bordism classes of the pseudocycle $H_0^\Psi(M)$ to the integers. Say that for a bordism class $[V] \in H_0^\Psi(M)$ represented by $f : V \to M$, $P_i \subset V$ is the submanifold consisting of those points in V that are mapped into M_i by f. Then define

$$\begin{aligned}\phi_i([V]) = \#&(\text{points with positive orientation in } P_i)\\ -\#&(\text{points with negative orientation in } P_i).\end{aligned}$$

Obviously $\phi_i([V] \amalg [W]) = \phi_i([V]) + \phi_i([W])$. The fact that this homomorphism is well defined is best seen in a concrete example. Say P_i consists of 3 positively and 2 negatively oriented points. Then there is a 0-manifold, consisting of one positively oriented point $\tilde{p}$, which is bordant to these five points. It looks like this: Take the 1-manifold consisting of three copies of the (closed) unit interval, which is mapped into M_i in such a way that it takes the form of three paths, one from $\tilde{p}$ to one of the positively oriented points in the image of P_i, and two leading from a negatively oriented point in the image of P_i to one that is positively oriented. Then the boundary of this manifold will be $-\tilde{p} \amalg P_i$. So $\tilde{p}$ is bordant to P_i.

Figure 2



Bordism between P and $\tilde{p}$

This gives us for every path component M_i of M a homomorphism from $H_0^\Psi(M)$ to the integers. Since the only connected 1-manifold with two boundary components is the closed unit interval, it is easy to see that the combination of these homomorphisms is indeed an isomorphism from $H_0^\Psi(M)$ to $\bigoplus_{\pi_0(M)} \mathbb{Z}$. □

Remark. This already shows that $H_0^\Psi(M) \cong H_0(M;\mathbb{Z})$. The isomorphism holds in general in all dimensions, as demonstrated in [Zin18]: $H_d^\Psi(M) \cong H_d(M;\mathbb{Z})$. We will come back to this in section 3.4.

3.2 The Intersection Product

We now want to introduce a product on the pseudocycle groups $H_*^\Psi(M)$, the *intersection product.* To do this, we first have to talk about a property that a pair of two pseudocycles can have, namely that of *strong transversality* of pseudocycles:

Definition 3.5. *Two pseudocycles $f_1 : V_1 \to M$ and $f_2 : V_2 \to M$ are called* ***strongly transverse*** *if there exist maps $e_1 : U_1 \to M$ and $e_2 : U_2 \to M$ with $\Omega_{f_1} \subset e_1(U_1)$ and $\Omega_{f_2} \subset e_2(U_2)$ such that $dim(U_i) \leq dim(V_i) - 2$ for $i = 1, 2$ and*

$$f_1 \pitchfork f_2, \quad f_1 \pitchfork e_2, \quad e_1 \pitchfork f_2, \quad e_1 \pitchfork e_2.$$

The definition of the intersection product of two bordism classes of pseudocycles $[f_1]$ and $[f_2]$ is going to make sense only if we can find representatives that are strongly transverse, since we will need transversality to construct a new manifold that can serve as the domain for the new "product"-pseudocycle. The following two Lemmata guarantee that such representatives can always be found:

Lemma 3.4. *If $f_1 : V_1 \to M$ and $f_2 : V_2 \to M$ are two pseudocycles, then there is a comeager set $\mathrm{Diff}_{reg}(M, f_1, f_2) \subset \mathrm{Diff}(M)$, such that for all $\phi \in \mathrm{Diff}_{reg}(M, f_1, f_2)$, $f_1 : V_1 \to M$ is strongly transverse to $\phi \circ f_2 : V_2 \to M$.*

Proof. Since f_1, f_2 are pseudocycles, there are sets and maps $e_1 : U_1 \to M$ and $e_2 : U_2 \to M$ such that $dim(U_i) \leq dim(V_i) - 2$ and $\Omega_{f_i} \subset e_i(U_i)$ for $i = 1, 2$. We show that there are four comeager subsets D_1, D_2, D_3, D_4 of $\mathrm{Diff}(M)$ such that

$$\forall \phi \in D_1 : f_1 \pitchfork \phi \circ f_2, \quad \forall \phi \in D_2 : f_1 \pitchfork \phi \circ e_2, \tag{3.1}$$

$$\forall \phi \in D_3 : e_1 \pitchfork \phi \circ f_2 \quad \text{and } \forall \phi \in D_4 : e_1 \pitchfork \phi \circ e_2 \tag{3.2}$$

Then the intersection $D := \bigcap_{1\leq i\leq 4} D_i$ will also be comeager, and for all $\phi \in D$, f_1 and $\phi \circ f_2$ will be strongly transverse.

The proof of each of the four claims in 3.1 and 3.2 proceeds along similar lines, so we will discuss only the first one explicitly.

Consider the map

$$\begin{aligned} m : V_1 \times V_2 \times \mathrm{Diff}(M) &\to M \times M \\ (v_1, v_2, \phi) &\mapsto (f_1(v_1), \phi(f_2(v_2))). \end{aligned}$$

Unfortunately, Diff(M) can not be given the structure of a finite-dimensional smooth manifold, so we are a bit out of our comfort zone - to make this proof precise, we would have to talk about completions to Banach manifolds, which would demand a lengthy excursion into functional analysis way beyond the scope of this thesis. We'll just mention the following:

There is an infinite-dimensional version of the transversality theorem, which says

Theorem 3.5. *Suppose that $F : X \times S \to Y$ is a C^k map of C^∞-Banach manifolds. Assume that*

1. *X, S and Y are nonempty, metrizable C^∞-Banach manifolds with chart spaces over a field $\mathbb{K}$.*
2. *The C^k-map $F : X \times S \to Y$ with $k \geq 1$ has y as a regular value.*
3. *For each parameter $s \in S$, the map $f_s(x) = F(x, s)$ is a Fredholm map, where* ind $Df_s(x) < k$ *for every $x \in f_s^{-1}(\{y\})$.*
4. *The convergence $\lim_{n\to\infty} s_n = s$ on S and $F(x_n, s_n) = y$ for all n imply the existence of a convergent subsequence $x_n \to x$ with $x \in X$*

Then there exists an open, dense subset S_0 of S such that y is a regular value of f_s for each parameter $s \in S_0$. And for a fixed element $s \in S_0$, if there is $n \in \mathbb{N}_0$ with ind $Df_s(x) = n$ *for all $x \in X$ such that $f_s(x) = y$, then $f_s^{-1}(\{y\})$ is either an n-dimensional C^k-Banach manifold or empty.*[2]

Without going to much into detail, we will just assume that

$$\mathcal{M} := \{(v_1, v_2, \phi) \mid f_1(v_1) = \phi(f_2(v_2))\} \subset V_1 \times V_2 \times \mathrm{Diff}(M),$$

which is nothing else than the preimage of the diagonal $\Delta \subset M \times M$ under m, is a manifold[3]. Then Sard's theorem (in Smale's version, cf. [Sma65]), if applied to the restriction of the projection map

$$\pi : V_1 \times V_2 \times \mathrm{Diff}(M) \to \mathrm{Diff}(M)$$

to $\mathcal{M}$, tells us that almost every point in Diff(M) is a regular value of $\pi_{|\mathcal{M}}$. So for almost every $\phi \in \mathrm{Diff}(M)$,

$$\pi_{|\mathcal{M}}^{-1}(\phi) = (f_1, \phi(f_2))^{-1}(\Delta) \times \{\phi\},$$

which is clearly diffeomorphic to $(f_1, \phi(f_2))^{-1}(\Delta)$, is a manifold.[4] □

[2] Quoted here from [Wik]; a proof could be assembled from [Hir94] and [Sma65].
[3] This should be provable using Theorem 1.2 in Chapter XIV of [Lan93]
[4] This proof is a slight modification of the proof of Lemma 6.5.5 in [MS04].

For the intersection product, we are going to define a new pseudocycle from two strongly transverse pseudocycles, as described in the following Lemma:

Lemma 3.6. *Let $f_1 : V_1 \to M$ and $f_2 : V_2 \to M$ be two strongly transverse pseudocycles of dimensions d_1 and d_2, and let $dim(M) = n$.*

Then $V := (f_1, f_2)^{-1}(\Delta) \subset V_1 \times V_2$ (where $\Delta \subset M \times M$ is the diagonal) is a manifold without boundary of dimension $(d_1 + d_2 - n)$, and itself the domain of a pseudocycle $f : V \to M$, $[f] \in H^{\Psi}_{d_1+d_2-n}(M)$ where f is defined via $f(x_1, x_2) := f_1(x_1) = f_2(x_2)$. The bordism class of this pseudocycle depends only on the bordism classes of f_1 und f_2 .

Proof. Lemma 2.5 says that $P := (f_1, f_2)^{-1}(\Delta)$ is a manifold of dimension $d_1 + d_2 - n$ with empty boundary. The closure of the image of f is contained in the closure of the image of f_1, which is compact; so $\overline{f(P)}$ is also compact as a closed subset of a compact set.

It remains to be shown that Ω_f is of dimension at most $d_1 + d_2 - n - 2$.

Take a sequence $(x_n^{(1)}, x_n^{(2)})_{n\in\mathbb{N}}$ which does not converge in V, but for which $y = \lim_{n\to\infty} f(x_n^{(1)}, x_n^{(2)})$ in M exists.

Case 1: $x_n^{(i)}$ does not converge in V_i, but $x_n^{(j)}$ converges in V_j (where $i, j \in \{1, 2\}, i \neq j$. Without loss of generality assume $i = 1, j = 2$). Then $U_* := (e_1, f_2)^{-1}(\Delta)$ is a manifold of dimension at most

$$dim(U_1) + dim(V_2) - dim(M) \leq d_1 + d_2 - n - 2$$

and for the map $f^* : U_* \to M$, $f^*(x_1, x_2) = e_1(x_1) = f_2(x_2)$ we have $y \in f^*(U_*)$. So in this case $\dim \Omega_f \leq d_1 + d_2 - n - 2$ and we're done.

Case 2: $x_n^{(i)}$ does not converge in V_i and $x_n^{(j)}$ also does not converge in V_j. Then $y \in e_1(U_1) \cap e_2(U_2)$ and since $e_1 \overline{\pitchfork} e_2$, we can now define U_* to be $(e_1, e_2)^{-1}(\Delta)$. It will now be of dimension (d_1+d_2-n-4), and y will be contained in the image of the $(d_1 + d_2 - n - 4)$-dimensional manifold U_* under the map $f^*(x_1, x_2) = e_1(x_1) = e_2(x_2)$.

What still remains to be shown is that the result depends only on the bordism classes of the involved pseudocycles, so assume that $\tilde{f}_1 : \tilde{V}_1 \to M$ is bordant to $f_1 : V_1 \to M$ via a map $f_1^{(b)} : V_1^{(b)} \to M$ where $V_1^{(b)}$ is a $(d_1 + 1)$-dimensional manifold with $\partial V_1^{(b)} = V_1 \amalg -\tilde{V}_1$ and $f_1^{(b)}{}_{|V_1} = f_1$ and $f_1^{(b)}{}_{|\tilde{V}_1} = \tilde{f}_1$. Because of the proof of Lemma 3.4, we may assume $f_1^{(b)}$ and $\partial f_1^{(b)}$ to be transverse to f_2.

$V_1^{(b)} \times V_2$ is a manifold with boundary, with

$$\partial(V_1^{(b)} \times V_2) = V_1 \times V_2 \amalg -\tilde{V}_1 \times V_2$$

and

$$\tilde{V} := (f_1^{(b)}, f_2)^{-1}(\Delta) \subset V_1^{(b)} \times V_2$$

is a manifold with boundary of dimension $(d_1 + 1 + d_2 - n) = \dim V + 1$, and by Theorem 2.2,

$$\begin{aligned} \partial\tilde{V} &= \tilde{V} \cap \partial(V_1^{(b)} \times V_2) \\ &= \{(x, y) \in V_1^{(b)} \times V_2 \mid x \in (V_1 \amalg \tilde{V}_1) \wedge f_1^{(b)}(x) = f_2(y)\} \end{aligned}$$

now,

$$x \in V_1 \wedge f_1^{(b)}(x) = f_2(y) \iff x \in V_1 \wedge f_1(x) = f_2(y) \iff (x, y) \in (f_1, f_2)^{-1}(\Delta)$$

and

$$x \in \tilde{V}_1 \wedge f_1^{(b)}(x) = f_2(y) \iff x \in \tilde{V}_1 \wedge \tilde{f}_1(x) = f_2(y) \iff (x,y) \in (\tilde{f}_1, f_2)^{-1}(\Delta)$$

so

$$\partial \tilde{V} = (f_1, f_2)^{-1}(\Delta) \amalg -(\tilde{f}_1, f_2)^{-1}(\Delta),$$

and $\tilde{V}$, with the map $\tilde{f}'(x,y) = f_1^{(b)}(x)$ is a bordism between the two pseudocycles $(\tilde{f}_1, f_2)^{-1}(\Delta) \to M$ and $(f_1, f_2)^{-1}(\Delta) \to M$.

□

Definition 3.6. *For two strongly transverse pseudocycles $f_i : V_i \to M$ of dimension $d_i \geq 0$ ($i = 1, 2$) in an n-dimensional manifold M, their* ***intersection product***

$$f_1 \cdot f_2$$

is defined according to the construction in Lemma 3.6.

Remark. The intersection product on the set of pseudocycles induces a homomorphism

$$H^\Psi_{d_1}(M) \otimes H^\Psi_{d_2}(M) \to H^\Psi_{d_1+d_2-\dim M}(M) : [f_1] \otimes [f_2] \mapsto [f_1] \cdot [f_2] \tag{3.3}$$

according to the Lemma 3.6.

Lemma 3.4 tells us that this homomorphism does not depend on any assumptions about the transversality of the pseudocycles involved.

In the case $d_1 + d_2 = \dim M$, and if M is connected, the isomorphism $H^\Psi_0(M) = \mathbb{Z}$ mentioned earlier (Lemma 3.3) gives the **intersection number**

$$[f_1] \cdot [f_2] \in \mathbb{Z}.$$

The so defined intersection number has a natural geometric interpretation as the signed count of the points of intersection of two strongly transverse representatives $f_1 : V_1 \to M$ and $f_2 : V_2 \to M$, with the condition on the dimensions of Ω_{f_1} and Ω_{f_2} making sure that this count is finite and invariant under bordism as proven in Lemma 3.6.

Lemma 3.7. $([f_1] + [f_2]) \cdot [f_3] = [f_1] \cdot [f_3] + [f_2] \cdot [f_3]$

Proof. This is seen by writing out the definitions explicitly. □

3.3 Some Example Computations on $\mathbb{T}^2$

We now want to see these definitions in action, and so we will perform some computations with pseudocycles of dimension ≤ 1 on the torus $\mathbb{T}^2$, which will also demonstrate how the intersection product can be used to compute the pseudocycle groups.

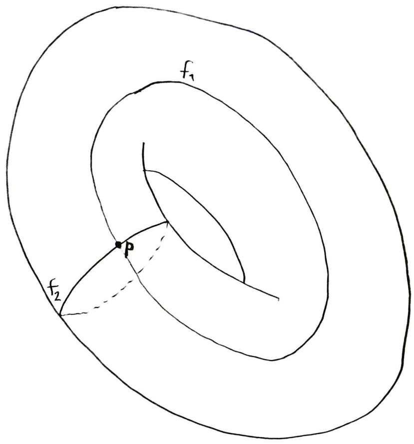
Since $\mathbb{T}^2$ is connected, we already know the zero-dimensional pseudocycle group: Lemma 3.3 tells us that $H^\Psi_0(\mathbb{T}^2) \cong \mathbb{Z}$, generated by any map $\{p\} \to \mathbb{T}^2$.

We are going to claim that $H^\Psi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$. and that the two generators are obtained by taking the two generators of the fundamental group $\pi_1(\mathbb{T}^2, p)$ (for a point $p \in \mathbb{T}^2$),

$$f_1 : (S^1, 1) \to (\mathbb{T}^2, p)$$
$$f_2 : (S^1, 1) \to (\mathbb{T}^2, p)$$

(as shown in figure 3) and ignoring the base points. (It is easily seen that the bordism classes of the pseudocycles so obtained do not depend on the chosen base point, since there exist obvious bordisms that can be used to move the pseudocycles into place.)

Figure 3



f_1 and f_2 in $\mathbb{T}^2$

So first, we are going to show that there can be no other generators in $H_1^\Psi(\mathbb{T}^2)$, and then that these two are in fact distinct, i.e. that there is no bordism between them.

The only one-dimensional manifolds without boundary modulo diffeomorphism are $(0, 1)$ and S^1.

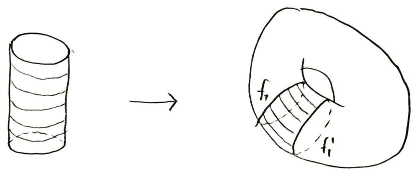
In order for any map $f : (0, 1) \to \mathbb{T}^2$ to be a pseudocycle, Ω_f would have to be empty (since manifolds of dimension $1 - 2 = -1$ are empty). But take a sequence (x_n) in $(0, 1)$ with $\lim_{n\to\infty} x_n = 0$. Since $\mathbb{T}^2$ is compact, $\lim_{n\to\infty} f(x_n)$ will exist in $\mathbb{T}^2$ and so Ω_f will be nonempty and of dimension 0. So the only 1-manifold that can serve as a domain for a pseudocycle in $\mathbb{T}^2$ is S^1.

Any map $f : (S^1, 1) \to (\mathbb{T}^2, p)$ which represents the neutral element in $\pi_1(\mathbb{T}^2)$, i.e. is homotopic to a constant map, will be bordant to the empty pseudocycle, since any homotopy $H : (S^1, 1) \times I \to (\mathbb{T}^2, p)$ with $H(x, 0) = f(x)$ and $H(x, 1) = p$ can be used to describe a bordism from $f : S^1 \to \mathbb{T}^2$ to the empty pseudocycle. H will in general be continuous but not necessarily smooth; but since $C_S^\infty(V, M)$ is dense in $C_S^0(V, N)$ (Theorem 2.6 in chapter 2 of [Hir94]), it can be perturbed by an arbitrarily small amount into a smooth homotopy, so we may assume H to be smooth. (The manifold of the bordism will be given by the manifold with boundary that is one hemisphere of S^2). So indeed there can be no generators in $H_1^\Psi(\mathbb{T}^2)$ other than f_1 and f_2.

In order to show that f_1 and f_2 do not represent the same element in $H_1^\Psi(\mathbb{T}^2)$, we are going to compute the intersection product of f_1 and f_2.

First, note that $[f_1] \cdot [f_1] = 0$. This is because there is a representative f_1' of $[f_1]$ such that $\operatorname{Im} f_1 \cap \operatorname{Im} f_1' = \emptyset$, as shown in figure 4

Figure 4



Bordism between f_1 and f_1'

Analogously, $[f_2] \cdot [f_2]$ is also 0.

To compute $[f_1] \cdot [f_2]$ and $[f_2] \cdot [f_1]$, we will have to think about orientations a little more; so let us consider $\mathbb{T}^2$ to be the product of two copies of S^1:

$$\mathbb{T}^2 = S^1_{(1)} \times S^1_{(2)}.$$

Say p has the coordinates $p = (p^{(1)}, p^{(2)})$ with $p^{(1)} \in S^1_{(1)}$ and $p^{(2)} \in S^1_{(2)}$; and say f_1 maps S^1 to $S^1_{(1)} \times \{p^{(2)}\}$ via $(e^{i\theta}) \mapsto (e^{i\theta}, p^{(2)})$ and similarly $f_2 : S^1 \to \{p^{(1)}\} \times S^1_{(2)}$ via $(e^{i\theta}) \mapsto (p^{(1)}, e^{i\theta})$.

Say furthermore that (e_θ) is a positively oriented basis of $T_{e^{i\theta}}(S^1)$, and define

$$v_1 := d(f_1)_1(e_0)$$
$$v_2 := d(f_2)_1(e_0)$$

Then (v_1) will be a positively oriented basis for $T_p(S^1_{(1)} \times \{p^{(2)}\})$ and (v_2) for $T_p(\{p^{(1)}\} \times S^1_{(2)})$, and (v_1, v_2) forms a basis for $T_p(\mathbb{T}^2)$ that is positively oriented according to the product orientation.

To get an orientation on $\Delta \subset \mathbb{T}^2 \times \mathbb{T}^2$, let $i : \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$; $x \mapsto (x, x)$ be the natural diffeomorphism from $\mathbb{T}^2$ to Δ, and define $\tilde{v}_j = di_p(v_j) = (v_j, v_j)$ for $j = 1, 2$. Then $(\tilde{v}_1, \tilde{v}_2)$ is a positively oriented basis for $T_{(p,p)}(\Delta)$.

We have: $(f_1, f_2)^{-1}(\Delta) = (f_1, f_2)^{-1}(p, p) = (1, 1)$.

So the domain of the pseudocycle $[f_1] \cdot [f_2]$ will be the single point $(1, 1) \in S^1 \times S^1$. To get its orientation, we look at the direct sum

$$d(f_1, f_2)_{(1,1)} T_{(1,1)}(S^1 \times S^1) \oplus T_{(p,p)}(\Delta) = T_{(p,p)}(\mathbb{T}^2 \times \mathbb{T}^2)$$

(cf. the paragraph on the preimage orientation in section 2.2.2, specifically the case where the preimage is of dimension zero.)

We have

$$d(f_1, f_2)_{(1,1)}(e_0, 0) = (v_1, 0)$$
$$d(f_1, f_2)_{(1,1)}(0, e_0) = (0, v_2)$$

The basis $((v_1, 0), (0, v_2), (v_1, v_1), (v_2, v_2))$ is negatively oriented as a basis of $T_{(p,p)}(\mathbb{T}^2 \times \mathbb{T}^2)$. This is so because the base transformation matrix from $((v_1, 0), (0, v_2), (v_1, v_1), (v_2, v_2))$ to the standard ordered basis of $T_{(p,p)}(\mathbb{T}^2 \times \mathbb{T}^2)$ in the product orientation, $((v_1, 0), (v_2, 0), (0, v_1), (0, v_2))$, is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which has determinant -1; so the orientation of $(1,1)$ as a 0-manifold is negative, and we get the pseudocycle

$$([f_1] \cdot [f_2]) : -(1,1) \to \mathbb{T}^2$$
$$(1,1) \mapsto p$$

It is now easy to see that $[f_2] \cdot [f_1]$ will be the same map, but with the orientation of the domain flipped; this is because while $d(f_1, f_2)_{(1,1)} : \mathbb{R}^2 \to \mathbb{R}^2$ is orientation preserving, $d(f_2, f_1)_{(1,1)}$ is orientation reversing, and thus $d(f_2, f_1)_{(1,1)} T_{(1,1)}(S^1 \times S^1)$ will have the opposite orientation of $d(f_1, f_2)_{(1,1)} T_{(1,1)}(S^1 \times S^1)$.

$$[f_2] \cdot [f_1] = -[f_1] \cdot [f_2]$$

Using this information about the intersection products, we can assure ourselves that $[f_1] - [f_2]$ is not zero in $H_1^{\Psi}(\mathbb{T}^1)$, because we can compute

$$([f_1] - [f_2]) \cdot ([f_1] + [f_2]) = 2[f_1] \cdot [f_2]$$

which is clearly not bordant to the empty pseudocycle (it is the disjoint union of two copies of the negatively oriented point $(1,1)$, as we have just discovered). But if $[f_1] - [f_2]$ were zero, then $([f_1] - [f_2]) \cdot ([f_1] + [f_2])$ would be zero also (see (3.3)).

So we have shown that $H_1^{\Psi}(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$.

□

3.4 Pseudocycles and Integral Homology

We now want to turn to a more elaborate investigation of the relationship between the pseudocycle groups and the integral homology groups. I have already given it away at the end of section 3.1 – they are isomorphic. The scope of this thesis will unfortunately not permit a full proof of this fact, for which one would have to turn to [Zin08]. We will, however, give a detailed description of the homomorphism that takes integral cycles to pseudocycles in section 3.4.1, and at least a brief sketch of other direction in section 3.4.2.

3.4.1 A Homomorphism $\Psi : H_d(M) \to H_d^{\Psi}(M)$

Theorem 3.8. *For every positive integer $d \geq 0$ there exists a natural homomorphism*

$$\Psi : H_d(M) \to H_d^{\Psi}(M)$$

such that

1. *For a pseudocycle $f : V \to M$ whose domain is a closed oriented manifold V, $\Psi(f_*[V]) = [f]$*
2. *For all $A, B \in H_*(M)$, $\Psi(A) \cdot \Psi(B) = \Psi(A \cdot B)$*

To make our life easier in proving this theorem, we are going to briefly talk about *smooth simplicial homology*:

Definition 3.7. *If M is a smooth manifold, a smooth d-simplex in M is a smooth map $\sigma : \Delta^d \to M$.*

Denote by $C_d^\infty(M)$ the subgroup of $C_d(M)$ (where $C_d(M)$ is the singular chain group of M, generated by the continuous singular d-simplices in M) that is generated by smooth simplices. Elements of this group are called ***smooth chains****. The boundary of a smooth simplex is a smooth chain, so we have a boundary map*

$$\partial : C_p^\infty(M) \to C_{p-1}^\infty(M)$$

and we can define the p^{th} ***smooth singular homology group*** *of M to be the quotient group*

$$H_p^\infty(M) := \ker(\partial_p)/\operatorname{Im}(\partial_{p+1})$$

(cf. [Lee93], p. 416 f.)

The following theorem will allow us to assume elements of $H_d(M)$ to be in $H_d^\infty(M)$ and thus to be represented by smooth cycles. We will state it without proof here – it can be looked up in [Lee93], where it is Theorem 16.6:

Theorem 3.9. *For any smooth manifold M, the map*

$$i_* : H_d^\infty(M) \to H_d(M)$$

induced by the inclusion is an isomorphism.

□

So for an element $A \in H_d(M)$, there is a smooth cycle $\sum_{i=1}^N c_i f_i$ such that $i_*^{-1}(A) \in H_d^\infty(M)$ is represented by $\sum_{i=1}^N c_i f_i$ according to Theorem 3.9, where $c_i \in \mathbb{Z}$ (we can assume without loss of generality that $c_i = \pm 1$) and $f_i : \Delta^d \to M$ are smooth maps. This will allow us to assume in our construction of Ψ that we are dealing with homology classes represented by smooth chains.

The proof of Theorem 3.8 will be divided into two Lemmata: First we will construct a d-dimensional pseudocycle from a smooth integral d-cycle, and then we will show that under this construction, homologous smooth cycles will lead to bordant pseudocycles.[5]

One more remark on notation: We will think of Δ^d as the convex hull of $(d+1)$ points $p_0, \dots, p_d$; and denote by $\Delta^d_{p_k}$ the $(d-1)$-simplex obtained by taking the convex hull of these same points with p_k removed.

Lemma 3.10. *If M is a smooth manifold, every integral d-cycle in $C_d^\infty(M)$ determines an element of $H_d^\Psi(M)$.*

Proof. First, let us introduce some sets associated to a singular d-chain $\gamma = \sum_{i=1}^N c_i f_i$. We define

$$S_\gamma := \coprod_{j=1}^N \{j\} \times c_i \Delta^d$$

[5] The proofs of these lemmata are adapted from [Zin08], where they are Lemma 3.2 and 3.3.

and

$$\tilde{f} : S_\gamma \to M$$
$$\tilde{f}(j,t) = f_j(t)$$

so that S_γ is the set of copies of the d-simplices that can be thought of as the domains of the maps f_i. Also, define

$$B_\gamma := \coprod_{\substack{0 \le j \le N \\ 0 \le k \le d}} \{j\} \times c_j(-1)^k \Delta^d_{p_k}$$

the disjoint union over the $(d-1)$-dimensional boundary faces of these simplices. Let $\pi_2 : B_\gamma \to \Delta^d$ be the projection onto the second coordinate.

In order to construct a pseudocycle, we want to turn the simplices in S_γ into smooth oriented manifolds with boundary and glue them together to form a smooth oriented manifold without boundary. The problem is that for $d \ge 2$, the points in the $(d-2)$-skeleton of Δ^d do not have neighborhoods diffeomorphic to an open subset of $\mathbb{H}^d$. On the upside, however, all points in Δ^d **not** in the $(d-2)$-skeleton are either in the interior of Δ^d or in the interior of the $(d-1)$-dimensional boundary faces, and have neighborhoods that are disjoint from the $(d-2)$-skeleton and straightforwardly diffeomorphic to open subsets of $\mathbb{H}^d$; so we get a smooth manifold by removing the $(d-2)$-skeleton from Δ^d.

Whether we remove the $(d-2)$-skeleton first, and glue afterwards – as is done in [Wen18] – or whether we glue first and then remove the $(d-2)$-skeleton as [Zin08] should make no difference for the resulting space. Here, we are going to take the latter route, which goes as follows:

If γ is a cycle, $\partial\gamma = \sum_{i=1}^N c_i \partial f_i = 0$, so the disjoint components of B_γ cancel in pairs, i.e. there is a continuous, orientation-reversing bijection

$$\varphi : B_\gamma \to B_\gamma$$

(which can also be viewed as a map $S_\gamma \to S_\gamma$ by a construction along the lines of [Zin08], p. 2747) such that for all $0 \le j \le N$ and $0 \le k \le d$, there are $0 \le l \le N$ and $0 \le m \le d$ such that:

$$\varphi((j, \Delta^d_{p_k})) = (l, \Delta^d_{p_m})$$

with φ mapping the boundary face $(j, \Delta^d_{p_k})$ to that which cancels it in $\sum_{i=1}^N c_i \partial f_i$ via the natural identification of $(d-1)$-simplices. Then

$$\varphi((j, \Delta^d_{p_k})) = (l, \Delta^d_{p_m}) \implies \left(\varphi((l, \Delta^d_{p_m})) = (j, \Delta^d_{p_k}) \wedge c_l = (-1)^{k+m+1} c_j\right)$$

Note that φ induces a smooth map on the interiors of the boundary faces.

Since φ identifies cancelling boundary faces, we have

$$\tilde{f}_{|B_\gamma} \circ \varphi = \tilde{f}_{|B_\gamma}$$

Now define the topological space

$$V' := \coprod_{j=1}^N \{j\} \times \Delta^d \Big/ ((j,t) \sim \varphi((j,t))) \, .$$

and let π be the quotient map

$$\pi : \coprod_{j=1}^{N} \{j\} \times \Delta^d \to V'. \tag{3.4}$$

Since it still contains the $(d-2)$-skeletons, V' is not smooth for the reasons described above; to get a smooth manifold, we put

$$V := V' - \pi \left(\coprod_{j=1}^{N} \{j\} \times Y \right)$$

where Y is the $(d-2)$-skeleton of Δ^d. Since V' is compact, $\Omega_{\tilde{f}} \subset \tilde{f}(V' - V)$. But $V' - V$ is the union of the $(d-2)$-skeletons, and thus has dimension $d-2$, which means that

$$\dim \Omega_{\tilde{f}} \leq d - 2. \tag{3.5}$$

We have to verify that the so constructed V is indeed a smooth oriented manifold. If $t \in \operatorname{int} \Delta^d$, then $\pi(\{j\} \times \operatorname{int} \Delta^d)$ is an open set around (j, t) which is naturally diffeomorphic to $\operatorname{int} \Delta^d$. So the critical points that need our special attention are those that are not in the interior of a d-simplex, and are of the form

$$[(j, t)] = [\varphi((j, t))]$$

with $t \in \operatorname{int}(\Delta^{d-1})$; Say t is in $\operatorname{int} \Delta^d_{p_1}$ and $\varphi(\{j\} \times \Delta^d_{p_1}) = \{l\} \times \Delta^d_{p_2}$ (in this case $\pi_2(\varphi((j,t)) \in \operatorname{int} \Delta^d_{p_2})$. Then take open neighborhoods $U^d_{p_1} \subset \Delta^d$ of $\operatorname{int} \Delta^d_{p_1}$ in Δ^d (and analogously $U^d_{p_2} \supset \operatorname{int} \Delta^d_{p_2}$) like described in [Zin08], section 2.1. Let

$$U := \pi\left(\{j\} \times U^d_{p_1}\right) \cup \pi\left(\{l\} \times U^d_{p_2}\right).$$

Then U is an open neighborhood of $[(j,t)] = [\varphi((j,t))]$ in V, which is homeomorphic to the disjoint union $U^d_{p_1} \amalg U^d_{p_2}$ with $\operatorname{int}(\Delta^d_{p_1}) \subset U^d_{p_1}$ and $\operatorname{int}(\Delta^d_{p_2}) \subset U^d_{p_2}$ identified via φ. So U is an open subset of $\mathbb{R}^k$ (cf. also [Zin08], part (2) of the proof of Lemma 3.2). So we get a natural coordinate chart for U, and for any overlap with another, the transition map will be the identity map on an open subset of $\operatorname{int} \Delta^d$.

Since φ is orientation reversing, the induced orientations on the tangent spaces $T_p(U)$ coming from the two d-manifolds with boundary $U^d_{p_1}$ and $U^d_{p_2}$ will agree. So V is indeed a smooth oriented manifold.

How about $\tilde{f}$, is it smooth? Well, it is continuous for sure (cf. Theorem 22.2 in [Mun00]). So we can use the fact that $C^\infty_S(V, M)$ is dense in $C^0_S(V, N)$ (Theorem 2.6 in chapter 2 of [Hir94]) to get a C^∞-map $f : V \to M$ that is arbitrarily close to $\tilde{f}$ in the strong topology $C^0_S(V, M)$. In particular, f can be chosen so as to have the same omega-limit set as $\tilde{f}$, which by (3.5) completes the proof that $f : V \to M$ is indeed a pseudocycle.

□

Lemma 3.11. *Under the construction of Lemma 3.10, homologous d-cycles give bordant pseudocycles.*

Proof. Say we have smooth chains

$$\gamma_0 = \sum_{i=1}^{N_0} c_i^{(0)} f_i^{(0)} \in C_d^\infty(M), \quad \gamma_1 = \sum_{i=1}^{N_1} c_i^{(1)} f_i^{(1)} \in C_d^\infty(M)$$
$$\text{and} \quad \gamma_2 = \sum_{i=1}^{N_2} c_i^{(2)} f_i^{(2)} \in C_{d+1}^\infty(M)$$

such that

$$\partial\gamma_2 = \gamma_1 - \gamma_0. \tag{3.6}$$

Let $(V_i', V_i, f^{(i)})), i = 0, 1$ be the manifolds and maps that the construction in Lemma 3.10 associates to γ_0 and γ_1.

We would love it if the manifold $\tilde{V}^*$ obtained by performing the constructions of Lemma 3.10 on γ_2 would already be the desired bordism between V_0 and V_1; unfortunately, though, things are not *quite* as simple. The boundary of $\tilde{V}^*$ is not $V_1 \amalg -V_0$. For if we drop the $((d+1)-2)$-dimensional boundary faces from the simplices in the domain of γ_2's maps, the boundary

$$\partial\tilde{V}^* = \tilde{V}_1 \amalg -\tilde{V}_0$$

will be lacking the $(d-1)$-dimensional boundary faces of the simplices that we glue together to get V_1 and V_0. To mitigate this, we will form a new manifold $\tilde{V}$ by attaching two collars,

$$\tilde{V}_0 \subset [0,1] \times V_0 \quad \text{and} \quad \tilde{V}_1 \subset [0,1] \times V_1,$$

to $\tilde{V}^*$ along its boundary components, $\tilde{V}_0$ and $\tilde{V}_0$.

This is how we do it, in detail:

Because of (3.6), the set B_{γ_2} assembling the boundary faces of the simplices that make up the domains of the maps in γ_2 can be partitioned into three sets:

First, a set $B^*_{\gamma_3}$ such that there exists $\varphi^* : B^*_{\gamma_2} \to B^*_{\gamma_2}$ like the φ described in the proof of Lemma 3.10, mapping cancelling boundary faces to each other in a way that restricts to a diffeomorphism on their interiors.

Then, two sets $B^{(i)}_{\gamma_2}, i = 0, 1$, defined via

$$\hat{B}^{(i)}_{\gamma_2} := \{\{j\} \times \Delta^{d+1}_{p_k} \subset B_{\gamma_2} \mid \exists(\{l\} \times \Delta^d) \in S_{\gamma_i} : f_l^{(i)} = f^{(2)}_{j}{}_{|\Delta^d_{p_k}} \text{ and } c_j^{(2)} = (-1)^k c_l^{(i)}\}$$
$$B^{(i)}_{\gamma_2} := \coprod_{\{j\} \times \Delta^{d+1}_{p_k} \in \hat{B}^{(i)}_{\gamma_2}} \{j\} \times \Delta^{d+1}_{p_k}$$

$\hat{B}^{(i)}_{\gamma_2}$ is the set of boundary faces of smooth singular simplices in γ_2 that are identified with smooth singular simplices in γ_i. And $B^{(i)}_{\gamma_2}$ is simply the disjoint union over the sets that are the elements of $\hat{B}^{(i)}_{\gamma_2}$.

There are maps $\varphi^{(i)} : S_{\gamma_i} \to B^{(i)}_{\gamma_2}, i = 0, 1$ which take $\{j\} \times \Delta^d$ to that simplex in $B^{(i)}_{\gamma_2}$ which is its appearance in $\partial\gamma_2$ (these maps exist because $\gamma_1 - \gamma_0 = \partial\gamma_2$)

Now, define

$$\tilde{V}' := \left(\coprod_{j=1}^{N_2} \{j\} \times \Delta^{d+1} \amalg \coprod_{i=0,1} \{i\} \times I \times V_i' \right) \Big/ \sim \text{, where} \tag{3.7}$$

$$(j,t) \sim \varphi * ((j,t)) \quad \forall j \in \{1, \ldots, N_2\}, t \in \Delta^{d+1} \tag{3.8}$$

$$(i, 1-i, \pi(k,t)) \sim \varphi^{(i)}(k,t) \quad \forall i \in \{0,1\}, 1 \leq k \leq N_i, t \in \Delta^d \tag{3.9}$$

(where π is the projection map (3.4))
and let

$$\tilde{\pi} : \coprod_{i=1}^{N_2} \{j\} \times \Delta^{d+1} \amalg \coprod_{i=0,1} \{i\} \times I \times V_i' \to \tilde{V}'$$

be the quotient map. Now let

$$\tilde{V} := \tilde{V}' - \tilde{\pi} \left(\coprod_{j=1}^{N_2} \{j\} \times \tilde{Y} \amalg \coprod_{i=0,1} \{i\} \times I \times (V_i' - V_i) \right)$$

where $\tilde{Y}$ is the $(d-1)$-skeleton of the $(d+1)$-simplex Δ^{d+1} and $(V_i' - V_i)$ is the union of the $(d-2)$-skeletons of the simplices that were glued together to make V_i'. Note that because of the identification (3.9), the copy of V_i' that is found at $\{i\} \times \{1-i\} \times V_i'$ loses not only its $(d-2)$-skeleton, but its $(d-1)$-skeleton.

The map $\tilde{f} : \tilde{V} \to M$ shall now be given by the disjoint union of the maps $f_j^{(2)} : \Delta^{d+1} \to M$ on $\tilde{V} \cap \left(\coprod_{j=1}^{N_2} \{j\} \times \Delta^{d+1} \right) \Big/ \sim$, and by $\tilde{f}((i,s,x)) = f^{(i)}(x)$ on the rest of $\tilde{V}$. That this map is well defined and continuous is verified analogously to the proof of Lemma 3.10.

To see that $\tilde{V}$ is a smooth oriented manifold with $\partial \tilde{V} \approx V_1 - V_0$, consider how it is the product of gluing together three components:

The space

$$\tilde{V}^* := \tilde{\pi} \left(\coprod_{j=1}^{N_2} \{j\} \times (\Delta^{d+1} - \tilde{Y}) \right)$$

is a smooth oriented manifold by the same arguments as in the proof of Lemma 3.10, and its boundary is

$$\partial \tilde{V}^* \approx \coprod_{i=0,1} B_{\gamma_2}^{(i)}$$

For $i = 0, 1$, the space

$$\tilde{V}_i := \tilde{V} \cap \tilde{\pi}(\{i\} \times I \times V_i') \approx I \times V_i - \{1-i\} \times \bigcup_{j=1}^{N_i} \pi(\{j\} \times (\Delta^d - \operatorname{int} \Delta^d))$$

(subtracting $(\Delta^d - \operatorname{int} \Delta^d)$ at one end is the same as removing the $(d-1)$-skeleton), is a smooth oriented manifold with boundary by the proof of Lemma 3.10; and its boundary is

$$\partial \tilde{V}_i \approx (-1)^{i+1}(\{i\} \times V_i) \amalg (-1)^i \left(\{1-i\} \times \bigcup_{j=1}^{N_i} \{j\} \times \operatorname{int} \Delta^d \right)$$

with $\tilde{f}$ restricting to a smooth map on $\tilde{V}_i$ and on $\tilde{V}^*$.

$\tilde{V}$ is obtained by gluing these three $(d+1)$-manifolds together along components of their boundaries via the map

$$\{1-i\} \times \bigcup_{j=1}^{N_i} \{j\} \times \operatorname{int} \Delta^d \to B_{\gamma_2}^{(i)}$$
$$((1-i), j, t) \mapsto \varphi^{(i)}(j, t)$$

Since this overlap map is orientation reversing (and, again, by similar arguments as in the proof of Lemma 3.10), this identification produces a smooth oriented manifold, whose boundary is

$$\partial \tilde{V} = \tilde{\pi}(\{1\} \times \{0\} \times V_1) \amalg -\tilde{\pi}(\{0\} \times \{1\} \times V_0) \approx V_1 \amalg -V_0$$

□

The map $\Psi : H_d(M) \to H_d^{\Psi}(M)$ defined according to Lemma 3.10 is clearly a homomorphism: In both the Pseudocycle and Homology Groups, addition can be viewed as taking the disjoint union.

Now, let's prove the properties (1) and (2) from Theorem 3.8[6].

Assume $A \in H_d(M)$ can be realised as $A = f_*[V]$, where $[V]$ is the fundamental class of a d-dimensional closed oriented manifold and $f : V \to M$ is smooth. Then what is $\Psi(f_*[V])$?

We know that V, since it is a smooth manifold, admits a triangulation; so we can view it as a simplicial complex, and its fundamental class as being represented by a smooth singular cycle $\sum c_i g_i$ with $c_i = \pm 1$ and g_i being a diffeomorphism from Δ^d to one of the simplices in said simplicial complex for every i.

Then $f_*[V] \in H_d(M)$ will be represented by $\sum c_i(f \circ g_i)$, and to calculate $\Psi(f_*[V])$, we will, as described in the proof of Lemma 3.10, glue together a manifold $\tilde{V}$ based on $\sum c_i(f \circ g_i)$. The so glued manifold can be naturally identified with the complement of V's $(d-2)$-skeleton in V, and we get a pseudocycle

$$\tilde{f} = f_{|\tilde{V}} : \tilde{V} \to M$$

which we can recognize to be bordant to the pseudocycle $f : V \to M$ if we use the collar-gluing technique from the proof of Lemma 3.11 to glue collars $[-1, 0] \times \tilde{V}$ and $(0, 1] \times V$ together.

So

$$\Psi(f_*[V]) = [\tilde{f}] = [f]$$

which concludes the proof that Ψ has property (1).

For property (2), recall that Thom's Theorem (as stated for example in footnote 36 on page 186 of [Wen19]) says that for any class $A \in H_d(M)$, there exist a closed oriented d-manifold V, a smooth map $f : V \to M$ and a positive integer $m \in \mathbb{Z}^+$ such that

$$mA = f_*[V]$$

So let $A, B \in H_*(M)$ be homology classes, and choose suitable integers, manifolds and maps such that

$$mA = f_*[V]$$
$$nB = g_*[W].$$

[6] These proofs follow p. 152 f. in [Wen18].

Then, using bilinearity of the intersection product, and the fact that Ψ is a homomorphism, we get

$$\Psi(A \cdot B) = \frac{\Psi(nA \cdot mB)}{nm} = \frac{\Psi(f_*[V] \cdot g_*[W])}{nm}.$$

With some standard results from intersection theory (§VI.11 in [Bre03], cf. [Wen18], p. 152) we can go on to assert:

$$= \frac{[f] \cdot [g]}{nm} = \frac{\Psi(f_*([V])) \cdot \Psi(g_*([W]))}{nm} = \frac{\Psi(mA) \cdot \Psi(nB)}{nm} = \Psi(A) \cdot \Psi(B)$$

□

3.4.2 Outlining the Idea of a Homomorphism $\Phi : H_d^\Psi(M) \to H_d(M)$

We will conclude this thesis by briefly talking about the homomorphism in the other direction, taking pseudocycles to integral cycles. We will not discuss a thorough proof that this homomorphism is well defined, as we did for Ψ in the previous section, but confine ourselves to trying to get an intuition of the idea behind the construction of this homomorphism, and paint both the construction and the proof of its well-definedness in rather broad strokes. A more detailed and rigorous discussion of this topic can be found in [Zin08], section 3.2; Zinger also proves that the compositions of Φ and Ψ are the respective identity maps, and that $H_*^\Psi(M)$ and $H_*(M)$ are thus indeed isomorphic.

One proposition that is necessary for the construction of the homomorphism from $H_d^\Psi(M)$ to $H_d(M)$ is the following one in [Zin08], where can be found as Proposition 2.2:

Proposition 3.12. *If $h : X \to Y$ is a smooth map and W is an open neighborhood of a subset A of $\mathrm{Im}(h)$ in Y, there exists a neighborhood U of A in W such that*

$$H_l(U) = 0 \quad \textit{if } l > \dim X$$

This is useful in the construction of integral cycles from pseudocycles, since it tells us that, if we have a $d-$pseudocycle $f : V \to M$ and its associated map $e : N \to M$ where $\dim(N) \leq d - 2$ and $\Omega_f \subset e(N)$, we get for any open neighborhood W of $\Omega_f \subset e(N)$ in M a neighborhood $U \subset W$ of Ω_f such that

$$H_l(U) = 0 \quad \text{if } l > d - 2.$$

That is, a neighborhood U of Ω_f such that the long exact sequence of the pair (M, U),

$$\ldots \to \underbrace{H_d(U)}_{=0} \xrightarrow{i_*} H_d(M) \xrightarrow{j_*} H_d(M, U) \xrightarrow{\partial} \underbrace{H_{d-1}(U)}_{=0} \to \ldots,$$

gives us a natural isomorphism

$$j_* : H_d(M) \to H_d(M, U) \tag{3.10}$$

that will turn out to be useful later on.

We define a subset $K = V - f^{-1}(U)$; K will be a compact subset of V. Let W be an open neighborhood of K in V such that $\overline{W}$ is a compact manifold with boundary. Then $\partial\overline{W}$ is contained in U. $\overline{W}$ inherits an orientation from V, so we get a fundamental class

$$[\overline{W}] \in H_d(\overline{W}, \partial\overline{W})$$

Furthermore, $f_{|\overline{W}} : (\overline{W}, \partial\overline{W}) \to (M, U)$ is a map of pairs and induces a homomorphism on homology

$$f_* : H_d(\overline{W}, \partial\overline{W}) \to H_d(M, U) \tag{3.11}$$

and we will set

$$[f] = f_*[\overline{W}] \in H_d(M, U) \cong H_d(M)$$

It remains to be shown that the homology class $[f]$ is independent of the choice of W, independent of the choice of U and that two bordant pseudocycles determine the same homology class.

To see that $[f]$ is independent of the choice of W, let W' be another choice such that $\overline{W} \subset W'$. For a triangulation on $(\partial\overline{W}) \cup (\partial\overline{W}')$, there exists a triangulation of $\overline{W}'$ which extends that triangulation (according to §16 in [Mun84]). Thus we see that the two cycles $f_*[\overline{W}]$ and $f_*[\overline{W}']$ in $H_d(M, U)$ will differ only by singular simplices lying in U, and thus determine the same element in $H_d(M, U)$.

If $U' \subset U$ is another choice of the set U, then the isomorphism (3.10) can be split into two isomorphisms

$$H_d(M) \xrightarrow{j_*^{(1)}} H_d(M, U') \xrightarrow{j_*^{(2)}} H_d(M, U)$$

and the homomorphism (3.11) is the composition

$$H_d(\overline{W}, \partial\overline{W}) \xrightarrow{f'_*} H_d(M, U') \xrightarrow{j_*^{(2)}} H_d(M, U)$$

So the homology class we get in $H_d(M)$ if we choose U' is the same one that we get if we choose U.

Now, let

$$f_0 : V_0 \to M, \quad f_1 : V_1 \to M$$

be two bordant pseudocycles and

$$\tilde{f} : \tilde{V} \to M$$

the bordism between them, i.e. $\partial\tilde{V} = V_1 \amalg -V_0$ and $\tilde{f}_{|V_i} = f_i$ for $i = 0, 1$. Then we get, as discussed, open neighborhoods of their respective omega limit sets, $\Omega_{f_i} \subset U_i$ and $\Omega_{\tilde{f}} \subset \tilde{U}$ such that

$$H_l(\tilde{U}) = 0 \quad l > d - 1$$
$$H_l(U_i) = 0 \quad l > d - 2.$$

We then learn from the long exact sequence of $H_*(M, \tilde{U})$,

$$\ldots \to \underbrace{H_d(\tilde{U})}_{=0} \to H_d(M) \to H_d(M, \tilde{U}) \to \ldots,$$

that the map $H_d(M) \to H_d(M, \tilde{U})$ is injective. It is equal to the composites

$$H_d(M) \to H_d(M, U_0) \xrightarrow{i_*^{(0)}} H_d(M, \tilde{U})$$
$$H_d(M) \to H_d(M, U_1) \xrightarrow{i_*^{(1)}} H_d(M, \tilde{U})$$

So if the homology classes $[f_0], [f_1]$, mapped from $H_d(M, U_i)$ into $H_d(M, \tilde{U})$ by the homomorphism induced by inclusion, are equal, this will tell us that they are equal in $H_d(M)$ and we are done.

The proof that $i_*^{(0)}[f_0]$ and $i_*^{(1)}[f_1]$ are equal in $H_d(M, \tilde{U})$ relies on the fact that it is possible to choose a triangulation of $\tilde{V}$ which extends some triangulations on its boundary components $V_i, i = 0, 1$ that in turn extend orientations on the sets $\partial\overline{W}_i$, such that subcomplexes triangulating $\overline{W}_1$ and $\overline{W}_2$ are the boundary of a finite subcomplex containing $\tilde{V} - \tilde{f}^{-1}(\tilde{U})$ in its interior. A detailed and rigorous discussion of this would demand a lot more algebraic topology, for which this thesis is not the proper place. What is the proper place for it is [Zin08], where it can be found in sections 2.3 and 3.2.

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